CENTRAL LIMIT THEOREM FOR RANDOM POLYMERS IN WEAK DISORDER

by

Tony Lam

A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

The University of Utah

August 2016

The University of Utah Graduate School

STATEMENT OF DISSERTATION APPROVAL

The dissertation of **Tony Lam**

has been approved by the following supervisory committee members:

Firas Rassoul-Agha ,	Chair(s)	**June 3, 2016** Date Approved
Alla Borisyuk ,	Member	**June 3, 2016** Date Approved
Davar Khoshnevisan ,	Member	**June 3, 2016** Date Approved
Andjres Treibergs ,	Member	**June 3, 2016** Date Approved
Timo Seppäläinen ,	Member	**June 3, 2016** Date Approved

by **Peter Trapa** , Chair/Dean of

the Department/College/School of **Mathematics**

and by **David B. Kieda** , Dean of The Graduate School.

ABSTRACT

We consider a d-dimensional random polymer measure in weak disorder. The potential depends on the site and the next step of the path. We prove an invariance principle for the quenched measure, in probability under the environment measure.

In loving memory of my grandparents Hung Chi Lam, So Yiu Lam, Francizek Odziemek, and Katherina Elizabeth Ingeborg Bolte Odziemek.

CONTENTS

ACKNOWLEDGEMENTS

Firstly, I would like to express my deepest gratitude to my advisor, Firas Rassoul-Agha, for his support during my Ph.D. study. His friendliness helped me persevere during a very dark time, and his patience deserves recognition as truly exceptional. Such great compassion meets with his great knowledge and wisdom. This work would not exist without him.

I would next like to thank my thesis committee members: Alla Borisyuk for reminding me of the importance of mathematics in the larger communities in science; Davar Khoshnevisan for reminding me of the importance and wisdom of perspective; Andrejs Treibergs for reminding me of the right time to be less technical and speak like a person; and Timo Seppäläilinen for asking hard questions seeking the best mathematical practices.

I'd like to thank Dr. Steven Edgley, Dr. Jennifer Majersik, Dr. Frances Harris, Mr. Sid Davis, and Dr. Karl Schwede for various student services, advocacy, and enrollment needs.

Lastly, I give thanks to my friends, my lifelines, for their support: Adam Wilson, Max Aeschbacher, Nathan Ballard, Tyson Chaplin, and Veronica Argyle; I thank my mom and dad for their support, and my older sister Brandi for coming home from school and showing her toddler brother this cool thing called math.

CHAPTER 1

INTRODUCTION

In this work, we study a generalization of a random walk process called the *directed polymer model*. Namely, we consider a random walk interacting with a random environment. The random environment represents the complex medium through which the random walk is traveling. This allows for a more realistic model that takes into account the fact that materials are not necessarily homogeneous. Mathematically, this inclusion of a second layer of randomness creates new difficulties and unexpected behaviors.

An important question, for instance, is whether or not the aforementioned diffusive behavior of the walk is affected by the inclusion of the random medium. We begin with some background.

1.1 Background

Random walk is arguably one of the most studied processes in probability theory. Besides its theoretical importance, random walk has been used to model a large variety of phenomena in statistical mechanics, biology, geophysics, climate studies, finance, social sciences, etc. One of the main and most often used results about random walk is the central limit theorem, which is a result that describes the diffusive behavior of the random walk. In general terms, the central limit theorem says that by time t, the "randomness" of the random walk is of order $\sqrt{t}$ and its "statistics" are close to Gaussian. Results on the central theorem date back to Laplace and de Moivre.

The directed polymer model was introduced in the statistical physics literature by Huse and Henley in 1985 [5] to study impurity-induced domain-wall roughening in the two-dimensional Ising model. The first rigorous mathematical study of directed polymers was done by Imbrie and Spencer [6] in 1988. Using an elaborate expansion, they proved that if the space dimension is three or larger and if the environment randomness is small, then the walk remains diffusive. Bolthausen [1] used a martingale technique to strengthen the result to a central limit theorem, for almost every realization of the random medium.

These results came as a surprise to the physics community that believed that directed polymers should always be superdiffusive (i.e., have fluctuations larger than those of a standard random walk). What is currently expected is that the polymer is indeed superdiffusive when the space dimension is one or two or when the the space dimension is three, but the randomness of the environment is large enough; see [2] and [8]. It is also believed that in one spatial dimension, the polymer fluctuates on the order of $t^{2/3}$ by time t. This has been shown to hold for Seppäläinen's log-gamma polymer model [14], where certain (hard) explicit computations are possible. See also [9–12, 16, 17].

1.2 Definitions and Notation

For a sequence (a_i), and integers $-\infty \le i \le j \le \infty$, let $a_{i,j} = (a_i, \dots, a_j)$. Let the d-dimensional canonical basis of $\mathbb{R}^d$ be denoted by $\mathcal{R} = \{e_1, \dots, e_d\}$. We will denote paths by (x_i) and random paths by (X_i). Our paths will be in $\mathbb{Z}^d$ and will have increments $z_{i+1} = x_{i+1} - x_i \in \mathcal{R}$. Similarly, the random increments are $Z_{i+1} = X_{i+1} - X_i$. We will denote the set of nonnegative integers by $\mathbb{Z}_+$, nonpositive integers by $\mathbb{Z}_-$, and positive integers by $\mathbb{N}$. Scalar product of $x, y \in \mathbb{R}^d$ will be denoted by $x \cdot y$. Set $\hat{u} = e_1 + \cdots + e_d$.

Each site $x \in \mathbb{Z}^d$ is assigned a random weight $\omega_x \in \mathbb{R}$. These weights are i.i.d. and their distribution is denoted by $\mathbb{P}$. Expectation relative to $\mathbb{P}$ is denoted by $\mathbb{E}$. Let $\omega = \{\omega_x : x \in \mathbb{Z}^d\}$ and for $y \in \mathbb{Z}^d$ define the shift $T_y\omega$ by $(T_y\omega)_x = \omega_{x+y}$. Denote by $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ the space of lattice weights ω.

We are also given a potential $g(\omega_0, z)$, a measurable function from $\mathbb{R} \times \mathcal{R} \to \mathbb{R}$. Common examples of the potential are $g(\omega_0, z) = \omega_0$ and $g(\omega_0, z) = \omega_0 + h \cdot z$ for some $h \in \mathbb{R}^d$. A good example to keep in mind throughout this manuscript is the following.

Example 1. *Using a measurable bijection between $\mathbb{R}$ and $\mathbb{R}^d$, we can take $\omega_x = (\omega_x^1, \dots, \omega_x^d)$ to be $\mathbb{R}^d$-valued, for $x \in \mathbb{Z}^d$. Then let $g(\omega_0, e_i) = \omega_0^i$ for $i \in \{1, \dots, d\}$. This means we put the random weights on the edges instead of on the vertices of $\mathbb{Z}^d$.*

A path $x_{0,n}$ has energy

$$H_n^\omega(x_{0,n}) = \sum_{i=0}^{n-1} g(\omega_{x_i}, z_{i+1}).$$

Let $p(z)$ be a probability vector on $\mathcal{R}$, i.e. $p(z) \ge 0$ and $\sum_{z \in \mathcal{R}} p(z) = 1$. Let P denote the random walk on $\mathbb{Z}^d$ with transition kernel p. This is the polymer measure at an "infinite" temperature. We denote expectation relative to P by E.

Given the weights ω and a $\beta > 0$ (seen as inverse temperature) the quenched, length n polymer measure is

$$\mu_n^\omega(x_{0,n}) = \frac{e^{\beta H_n^\omega(x_{0,n})}}{\mathcal{Z}_n^\omega} P(x_{0,n}),$$

where the partition function $\mathcal{Z}_n^\omega$ is the normalizing constant

$$\mathcal{Z}_n^\omega = \sum_{x_{0,n}} e^{\beta H_n^\omega(x_{0,n})} P(x_{0,n}) = E\left[e^{\beta H_n^\omega(X_{0,n})}\right].$$

Remark 1. *μ_n^ω and $\mathcal{Z}_n^\omega$ depend on β, but β is fixed so we omit it from notation. We will also drop the ω from $\mathcal{Z}_n^\omega$ and write $\mathcal{Z}_n$.*

Remark 2. *Note that we can write $P(x_{0,n}) = \sum_{i=0}^{n-1} e^{\log p(z_{i+1})}$ and so if we replace $g(\omega_0, z)$ by $g(\omega_0, z) + \log p(z)$ then we can set $p(z) = 1/|\mathcal{R}|$.*

1.3 Previous Results

Our main interest is in the fluctuations of $x_{0,n}$ under the polymer measure μ_n^ω. Imbrie and Spencer [6] proved that for the case $g(\omega_0, z) = \omega_0 \in \{-1, 1\}, p(z) = 1/|\mathcal{R}|, \beta$ small, and $d \geq 4$, there exists $C \in (0, \infty)$ and $\alpha \in (0, 1)$ such that for $\mathbb{P}$-a.e. ω and all n

$$\left| E^{\mu_n^\omega}\left[\left|X_n - \frac{n\hat{u}}{d}\right|_2^2\right] - \frac{d-1}{d} n\right| \leq Cn^{1-\alpha}.$$

(Here, $|\cdot|_2$ is the Euclidean norm.)

Bolthausen [1] used a martingale approach to extend this to a central limit theorem (CLT) for X_n under μ_n^ω, for $\mathbb{P}$-a.e. ω. More precisely, Bolthausen showed that for $\mathbb{P}$-a.e. ω, the distribution of

$$\frac{X_n - (n\hat{u}/d)}{\sqrt{(d-1)n/(d^2)}} \tag{1.1}$$

under μ_n^ω converges weakly to a $(d-1)$-dimensional standard normal. See also [6, 15]. Comets and Yoshida [3] showed that this CLT (and in fact an invariance principal) holds whenever weak disorder is in force. Weak disorder roughly means that μ_n^ω is close to P. It holds, for example, when β is small enough. The next chapter provides a more precise definition. However, in [3] it is only shown that (1.1) holds in $\mathbb{P}$-probability.

Remark 3. *In [1, 3, 6, 15], X_n had step increments in $\{\pm e_1, \dots, \pm e_d\}$ and therefore, there was no need for a centering and the variance of each coordinate was given by n/d. In our case, X_n has increments in $\mathcal{R} = \{e_1, \dots, e_d\}$ and hence the $n\hat{u}/d$ centering and $(d-1)n/d^2$ variance terms.*

1.4 Our Result

We will extend Comets and Yoshida's result to the case of a potential that depends on the increment z. Thus, we have to modify the arguments in [3] accordingly. For example, when $g(\omega_0, z) = g(\omega_0)$, the so-called *annealed* polymer measure, given by

$$\begin{aligned} Q(x_{0,n}) &= \frac{\mathbb{E}\left[e^{\sum_{i=0}^{n-1} \beta g(\omega_{x_i})}\right]}{E\mathbb{E}\left[e^{\sum_{i=0}^{n-1} \beta g(\omega_{X_i})}\right]} \\ &= \frac{\mathbb{E}\left[e^{\beta g(\omega_0)}\right]^n P(x_{0,n})}{\mathbb{E}\left[e^{\beta g(\omega_0)}\right]^n} \\ &= P(x_{0,n}), \end{aligned}$$

does not depend on β and is simply equal to P.

In our setting, however, we have

$$\begin{aligned} Q(x_{0,n}) &= \frac{\mathbb{E}\left[e^{\sum_{i=0}^{n-1} \beta g(\omega_{x_i}, z_{i+1})}\right]}{E\mathbb{E}\left[e^{\sum_{i=0}^{n-1} \beta g(\omega_{X_i}, Z_{i+1})}\right]} \\ &= \frac{\prod_{i=0}^{n-1} \mathbb{E}\left[e^{\beta g(\omega_0, z_{i+1})}\right] p(z_{i+1})}{e^{n\lambda(\beta)}} \\ &= \prod_{i=0}^{n-1} q(z_{i+1}), \end{aligned}$$

where

$$q(z) = \mathbb{E}\left[e^{\beta g(\omega_0, z) - \lambda(\beta)}\right] p(z), z \in \mathcal{R}.$$

In other words, Q is still a random walk, but it depends on β and is not equal to P unless $\beta = 0$.

In particular, to get an invariance principle for the distribution of X_n under Q, we need to center X_n by

$$v = \sum_{z \in \mathcal{R}} z q(z)$$

and scale $X_n \cdot e_i$ by $\sqrt{q(e_i)(1 - q(e_i))n}$. Note that $v \cdot \hat{u} = 1$. Since $X_n \cdot \hat{u} = n$ has no fluctuations, we will only be interested in the spacial components. Let Γ be the diagonal $d \times d$ matrix with entries $\Gamma_{ii} = 1/\sqrt{q(e_i)(1 - q(e_i))}$.

Define the scaled process:

$$B_n(t) = \frac{\Gamma(X_{\lfloor nt \rfloor} - \lfloor nt \rfloor v)}{\sqrt{n}}, 0 \le t \le 1.$$

Let $\mathbb{W}$ be the Wiener space of functions $B : [0,1] \to \hat{u}^{\perp} = \{x \in \mathbb{R}^d : x \cdot \hat{u} = 0\}$ that are right-continuous, have left limits, and satisfy $B(0) = 0$. Equip $\mathbb{W}$ with the topology induced by the sup norm. Let $C_b(\mathbb{W})$ be the space of bounded, continuous functions on $\mathbb{W}$.

Let $\mathbf{P}$ be a $\hat{u}$-valued standard Brownian motion. Denote by $\mathbf{E}$ the expectation relative to $\mathbf{P}$. Our main result is the following.

Theorem 1. *Assume weak disorder holds. Let $F \in C_b(\mathbb{W})$. Then $E^{\mu_n^\omega}[F(B_n)]$ converges in probability to $\mathbf{E}[F(B)]$.*

Remark 4. *Part of the proof of the version of this theorem in [3] relies on a martingale scheme that fails for the type of potential we consider. We thus modify the argument and avoid this martingale scheme. One consequence of our approach is that we first prove our result for an infinite-volume version of μ_n^ω and then use that result to get the result of Theorem 1.*

CHAPTER 2

WEAK AND STRONG DISORDER

Define the log-moment generating function,

$$\begin{aligned}\lambda(\beta) &= \log \mathbb{E}\left[\sum_{z\in\mathcal{R}} e^{\beta g(\omega_0,z)} p(z)\right] \\ &= \log \mathbb{E}E\left[e^{\beta g(\omega_0,Z_1)}\right].\end{aligned}$$

Lemma 1. *We have* $\mathbb{E}\mathcal{Z}_n^{\omega} = e^{n\lambda(\beta)}$.

Proof. Compute

$$\begin{aligned}\mathbb{E}\mathcal{Z}_n &= \mathbb{E}E\left[e^{\beta\sum_{i=0}^{n-1} g(\omega_{X_i}Z_{i+1})}\right] \\ &= E\Pi_{i=0}^{n-1}\mathbb{E}\left[e^{\beta g(\omega_0,Z_{i+1})}\right] \\ &= \Pi_{i=0}^{n-1}\mathbb{E}E\left[e^{\beta g(\omega_0,Z_1)}\right] \\ &= e^{n\lambda(\beta)}.\end{aligned}$$

Since all quantities are positive, we can apply Fubini's theorem (Appendix A.6, Theorem (6.2) page 470 of [4]) to interchange $\mathbb{E}$ and E. Then we use the fact that ω_{x_i} are i.i.d., then that Z_{i+1} are i.i.d. □

Define $\mathfrak{S}_n = \sigma(\omega_x : x\cdot\hat{u} \leq n)$, the σ-algebra generated by the weights ω_x below level n.

Lemma 2. $W_n = \frac{\mathcal{Z}_n}{\mathbb{E}\mathcal{Z}_n}$ *is a* $\mathfrak{S}_{n-1}$*-martingale.*

Proof. W_n is clearly $\mathfrak{S}_{n-1}$-measurable. Next write

$$
\begin{aligned}
\mathbb{E}[W_n|\mathfrak{S}_{n-2}] &= \frac{1}{e^{n\lambda(\beta)}}\mathbb{E}[\mathcal{Z}_n|\mathfrak{S}_{n-2}] \\
&= \frac{1}{e^{n\lambda(\beta)}} E\left[\mathbb{E}\left[e^{\beta\sum_{i=0}^{n-1} g(\omega_{X_i}, Z_{i+1})}\middle|\mathfrak{S}_{n-2}\right]\right] \\
&= \frac{1}{e^{n\lambda(\beta)}} E\left[e^{\beta\sum_{i=0}^{n-2} g(\omega_{X_i}, Z_{i+1})}\mathbb{E}\left[e^{\beta g(\omega_0, Z_n)}\right]\right] \\
&= \frac{1}{e^{n\lambda(\beta)}} E\left[e^{\beta\sum_{i=0}^{n-2} g(\omega_{X_i}, Z_{i+1})}\right] E\left[\mathbb{E}\left[e^{\beta g(\omega_0, Z_1)}\right]\right] \\
&= \frac{1}{e^{(n-1)\lambda(\beta)}} E\left[e^{\beta\sum_{i=0}^{n-2} g(\omega_{X_i}, Z_{i+1})}\right] \\
&= W_{n-1}.
\end{aligned}
$$

Interchanging expectations to get the first equality is allowed because all quantities are positive. The third equality comes because $e^{\beta\sum_{i=0}^{n-2} g(\omega_{X_i}, Z_{i+1})}$ is $\mathfrak{S}_{n-2}$ measurable, and also $e^{\beta g(\omega_{X_{n-1}}, Z_n)}$ is independent of $\mathfrak{S}_{n-2}$. $\square$

Since $W_n \geq 0$, the martingale convergence theorem (Chapter 4, Theorem 2.10, page 235 of [4]) assures the limit

$$W_\infty = \lim_{n\to\infty} W_n$$

exists $\mathbb{P}$-almost surely.

Lemma 3. *$\{W_\infty > 0\}$ is a tail event. That is, it is $\sigma(\omega_x : x \cdot \hat{u} \geq m)$-measurable for all m.*

Proof. Fix $m \in \mathbb{Z}_+$. Then

$$W_n = \frac{E\left[e^{\beta\sum_{i=0}^{m-1} g(\omega_{X_i}, Z_{i+1})} E\left[e^{\beta\sum_{i=m}^{n-1} g(\omega_{X_i}, Z_{i+1})}\middle|X_m\right]\right]}{e^{n\lambda(\beta)}}.$$

Since $e^{\beta\sum_{i=0}^{m-1} g(\omega_{x_i}, z_{i+1})}$ is always positive,

$$\{W_\infty = 0\} = \left\{\lim_{n\to\infty} \frac{E\left[e^{\beta\sum_{i=m}^{n-1} g(\omega_{X_i}, Z_{i+1})}\middle|X_m = x\right]}{e^{n\lambda(\beta)}} = 0,\ \forall x \in \mathbb{Z}_+^d \text{ with } x \cdot \hat{u} = m\right\}.$$

The claim follows since the expectation in the above display is $\sigma(\omega_x : x \cdot \hat{u} \geq m)$-measurable. $\square$

By Kolmogorov's 0-1 law (Chapter 1, Theorem 8.1, page 62 of [4]), we have that $\mathbb{P}(W_\infty > 0) \in \{0,1\}$. Based on this, Bolthausen [1] called the case when $W_\infty > 0$, $\mathbb{P}$-a.s. *weak disorder* and the *strong disorder* is when $W_\infty = 0$, $\mathbb{P}$-a.s. Weak disorder happens at high temperature (small β) and high dimension (d large).

Lemma 4. *If $d \geq 4$, then there exists a $\beta_0 > 0$ such that for $\beta \in [0, \beta_0)$, we have* $\mathbb{P}(W_\infty > 0) = 1$.

Recall random walk Q with kernel

$$q(z) = \mathbb{E}\left[e^{\beta g(\omega_0, Z) - \lambda(\beta)}\right] p(z).$$

Let $Q^{(2)}$ be the Markov chain $(X_n, \tilde{X}_n)$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ starting at $(0,0)$ with kernel

$$\begin{aligned} & q^{(2)}\left((x, \tilde{x}), (x+z, \tilde{x}+\tilde{z})\right) \\ &= \begin{cases} q(z)q(\tilde{z}), & \text{if } x \neq \tilde{x}, z, \tilde{z} \in \mathcal{R}, \\ \kappa(\beta)^{-1}\mathbb{E}\left[e^{\beta g(\omega_0, z) - \lambda(\beta)} e^{\beta g(\omega_0, \tilde{z}) - \lambda(\beta)}\right] p(z)p(\tilde{z}), & \text{if } x = \tilde{x}, z, \tilde{z} \in \mathcal{R}, \end{cases} \end{aligned}$$

where

$$\kappa(\beta) = E \otimes E\left[\mathbb{E}\left[e^{\beta g(\omega_0, Z_1) - \lambda(\beta)} e^{\beta g(\omega_0, \tilde{Z}_1) - \lambda(\beta)}\right]\right] = \mathbb{E}\left[\left(E\left[e^{\beta g(\omega_0, Z_1) - \lambda(\beta)}\right]\right)^2\right].$$

As long as $X_n \neq \tilde{X}_n$, the two evolve under $Q^{(2)}$ as independent Q-random walks. $Q^{(2)}$ differs from $Q \otimes Q$ only when $X_n = \tilde{X}_n$. Let $E^{(2)}$ denote expectation under $Q^{(2)}$.

Proof of Lemma 4. Start by computing

$$\begin{aligned} \mathbb{E}\left[W_n^2\right] &= \mathbb{E}\left[E \otimes E\left[e^{\sum_{i=0}^{n-1}\left(\beta g(\omega_{X_i}, Z_{i+1}) - \lambda(\beta)\right)} e^{\sum_{i=0}^{n-1}\left(\beta g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1}) - \lambda(\beta)\right)}\right]\right] \\ &= E \otimes E\left[\prod_{i=0}^{n-1} \mathbb{E}\left[e^{\beta g(\omega_{X_i}, Z_{i+1}) - \lambda(\beta)} e^{\beta g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1}) - \lambda(\beta)}\right]\right] \\ &= E \otimes E\left[\prod_{i=0}^{n-1} \frac{q^{(2)}\left((X_i, \tilde{X}_i), (X_{i+1}, \tilde{X}_{i+1})\right)}{p(Z_{i+1})p(\tilde{Z}_{i+1})} \kappa(\beta)^{\mathbf{1}\{X_i = \tilde{X}_i\}}\right] \\ &= E^{(2)}\left[\kappa(\beta)^{\sum_{i=0}^{n-1} \mathbf{1}\{X_i = \tilde{X}_i\}}\right]. \end{aligned}$$

Since $\kappa(\beta) > 1$, we see that $\mathbb{E}\left[W_n^2\right]$ is nondecreasing in n. Let $I_n = \sum_{i=0}^{n-1} \mathbf{1}\{X_i = \tilde{X}_i\}$ and $I_\infty = \sum_{i=0}^{\infty} \mathbf{1}\{X_i = \tilde{X}_i\}$. Then if

$$E^{(2)}\left[\kappa(\beta)^{I_\infty}\right] < \infty \tag{2.1}$$

we see that $\sup_n \mathbb{E}\left[W_n^2\right] \leq E^{(2)}\left[\kappa(\beta)^{I_\infty}\right] < \infty$ thus W_n is uniformly integrable (see [4], page 260). This implies that W_n converges to W_∞ also in $L^1(\mathbb{P})$ and $1 = \mathbb{E}[W_n] \to \mathbb{E}[W_\infty]$. This in turn implies that $\mathbb{P}(W_\infty > 0) > 0$ and thus that $\mathbb{P}(W_\infty > 0) = 1$ and we have weak disorder.

Next, we show that under the conditions of the lemma, (2.1) holds. To this end, observe that under $Q^{(2)}$ the process $X_n - \tilde{X}_n$ is itself a Markov chain $\overline{Q}$ with transition kernel

$$\overline{q}(u, u+v) = \begin{cases} \sum_{z-\tilde{z}=v} q(z)q(\tilde{z}) & \text{if } u \neq 0 \\ \sum_{z-\tilde{z}=v} q((0,0),(z,\tilde{z})) & \text{if } u = 0. \end{cases}$$

This is a symmetric $(d-1)$-dimensional random walk, perturbed at the origin. Consequently, under $Q^{(2)}$ the number of collisions I_∞ is a geometric random variable with success probability $\overline{Q}(\tau_0 < \infty)$ where τ_0 is the time of return to the origin. Note next that $q^{(2)} \to p \otimes p$ as $\beta \to 0$. This says that $\overline{Q} \to \overline{P}$, the difference of two independent P-random walks. Thus, as long as P is not deterministic (i.e., $p(z) \neq \delta_{z_0}(z)$ for some $z_0 \in \mathcal{R}$) we have

$$\lim_{\beta \to 0} \overline{Q}(\tau_0 < \infty) = \overline{P}(\tau_0 < \infty) < 1.$$

(Recall that $d \geq 4$ and thus the $(d-1)$-dimensional random walk $\overline{P}$ is transient.) On the other hand, if $s(\beta) = Q^{(2)}(\tau_0 < \infty)$, then

$$E^{(2)}\left[\kappa(\beta)^{I_\infty}\right] = \sum_{n=1}^{\infty} s(\beta)^{n-1}(1-s(\beta))\kappa(\beta)^n = \frac{(1-s(\beta))\kappa(\beta)}{1-s(\beta)\kappa(\beta)}.$$

So (2.1) holds as soon as $s(\beta)\kappa(\beta) < 1$. It is clear that $\kappa(\beta) \to 1$ as $\beta \to 0$ and we just showed that $s(\beta) \to \overline{P}(\tau_0 < \infty) < 1$ as $\beta \to 0$. Therefore, (2.1) holds for β small enough and we have argued above that this proves the lemma. $\square$

In this work, we are interested in the case of weak disorder. Hence, β will be chosen so that weak disorder holds, and then it is fixed. We will then write $g(\omega_0, z)$ for $\beta g(\omega_0, z) - \lambda(\beta)$, which means we can (and will) set $\beta = 1$ and $\lambda(\beta) = 0$. We will thus drop the dependence on β from $\kappa(\beta)$ and simply write κ.

CHAPTER 3

THE INFINITE POLYMER

Let us extend μ_n^ω to a measure on infinite polymer paths. Recall that polymer paths are encoded by their increments $(z_n)_{n\geq 1} \in \mathcal{R}^{\mathbb{N}}$. Let $\mathcal{F}$ be the Borel σ-algebra on $\mathcal{R}^{\mathbb{N}}$. For $A \in \mathcal{F}$, define the finite polymer measure

$$\mu_n^\omega(A) = \frac{E\left[e^{\sum_{i=0}^{n-1} g(\omega_{X_i}, Z_{i+1})}\mathbf{1}_A(X_{0,\infty})\right]}{\mathcal{Z}_n}.$$

In particular,

$$\mu_n^\omega(x_{0,n}) = \frac{e^{\sum_{i=0}^{n-1} g(\omega_{x_i}, z_{i+1})}}{\mathcal{Z}_n} P(x_{0,n})$$

as before, and

$$\mu_n^\omega(x_{0,m}) = \mu_n^\omega(x_{0,n}) P(x_{n,m}), \text{ if } m > n.$$

Lemma 5. *μ_n^ω is a Markov chain with transitions:*

$$\begin{aligned}
\mu_n^\omega(X_{m+1} = y | X_m = x) &= e^{g(\omega_x, y-x)} \frac{W_{n-m-1}(T_y\omega)}{W_{n-m}(T_x\omega)} p(y-x), && \textit{if } m < n \\
&= p(y-x). && \textit{if } m \geq n.
\end{aligned}$$

Proof. For $m < n$,

$$\begin{aligned}
&\mu_n^\omega(X_{m+1} = y | X_m = x, X_{0,m-1} = x_{0,m-1}) = \frac{\mu_n^\omega(x_{0,m-1}, x, y)}{\mu_n^\omega(x_{0,m-1}, x)} \\
&= \frac{e^{\sum_{i=0}^{m-1} g(\omega_{x_i}, z_{i+1})} P(x_{0,m-1}) p(x - x_{m-1}) e^{g(\omega_x, y-x)} p(y-x) E\left[e^{\sum_{i=m+1}^{n-1} g(\omega_{X_i}, Z_{i+1})} \middle| X_{m+1} = y\right]}{e^{\sum_{i=0}^{m-1} g(\omega_{x_i}, z_{i+1})} P(x_{0,m-1}) p(x - x_{m-1}) E\left[e^{\sum_{i=m}^{n-1} g(\omega_{X_i}, Z_{i+1})} \middle| X_m = x\right]} \\
&= e^{g(\omega_x, y-x)} \frac{W_{n-m-1}(T_y\omega)}{W_{n-m}(T_x\omega)} p(y-x),
\end{aligned}$$

and for $m \geq n$,

$$\mu_n^\omega(X_{m+1} = y | X_m = x, X_{0,m-1} = x_{0,m-1}) = \frac{\mu_n^\omega(x_{0,m-1}, x) p(y-x)}{\mu_n^\omega(x_{0,m-1}, x)} = p(y-x). \qquad \square$$

Theorem 2. *Assume weak disorder holds. For $\mathbb{P}$-almost every ω, μ_n^ω converges weakly to a limit μ^ω that is a Markov chain starting at 0 and using transition*

$$\mu^\omega(X_{m+1} = y | X_m = x) = e^{g(\omega_x, y-x)} \frac{W_\infty(T_y\omega)}{W_\infty(T_x\omega)} p(y-x).$$

Proof. Take $n \to \infty$ in Lemma 5. □

μ^ω is then called a random walk in the random environment ω.

Define $\mathfrak{S}_n^- = \sigma(\omega_x : x \cdot \hat{u} \geq -n)$, the σ-algebra generated by weights above level $-n$. Extend P to a two sided random walk $(X_n)_{n\in\mathbb{Z}}$ with $X_0 = 0$ and still using kernel p.

Lemma 6. $W_n^- = E\left[e^{\sum_{i=-n}^{-1} g(\omega_{X_i}, Z_{i+1})}\right]$ *is a $\mathfrak{S}_n^-$-martingale.*

Proof. W_n^- is $\mathfrak{S}_n^-$-measurable. Then we write

$$\begin{aligned}
\mathbb{E}\left[W_n^- \middle| \mathfrak{S}_{n-1}^-\right] &= \mathbb{E}\left[E\left[e^{\sum_{i=-n}^{-1} g(\omega_{X_i}, Z_{i+1})}\right] \middle| \mathfrak{S}_{n-1}^-\right] \\
&= \mathbb{E}\left[E\left[e^{g(\omega_{X_{-n}}, Z_{-n+1})} e^{\sum_{i=-(n-1)}^{-1} g(\omega_{X_{-n}}, Z_{-n+1})}\right] \middle| \mathfrak{S}_{n-1}^-\right] \\
&= E\left[\mathbb{E}\left[e^{g(\omega_{X_{-n}}, Z_{-n+1})}\right] e^{\sum_{i=-(n-1)}^{-1} g(\omega_{X_{-n}}, Z_{-n+1})}\right] \\
&= E\left[e^{\sum_{i=-(n-1)}^{-1} g(\omega_{X_{-n}}, Z_{-n+1})}\right] E\left[\mathbb{E}\left[e^{g(\omega_0, Z_1)}\right]\right] \\
&= W_{n-1}^-.
\end{aligned}$$

By the choice of $\lambda(\beta) = 0$, we have $E\left[\mathbb{E}\left[e^{g(\omega_0, Z_1)}\right]\right] = e^0 = 1$. □

The martingale convergence theorem implies that

$$W_\infty^- = \lim_{n\to\infty} W_n^-$$

exists $\mathbb{P}$-a.s.

Lemma 7. $\{W_\infty^- > 0\}$ *is a tail event.*

Proof. We show that $\{W_\infty^- = 0\}$ is $\sigma(\omega_x : x \cdot \hat{u} \leq -m)$-measurable for all m. Fix $m \in \mathbb{N}$. We write

$$W_n^- = E\left[e^{\sum_{i=-m}^{-1} g(\omega_{X_i}, Z_{i+1})} E\left[e^{\sum_{i=-n}^{-m-1} g(\omega_{X_i}, Z_{i+1})} \middle| X_{-m}\right]\right].$$

Since $e^{\sum_{i=-m}^{-1} g(\omega_{X_i}, Z_{i+1})}$ is always positive,

$$\{W_\infty^- = 0\} = \left\{\lim_{n\to\infty} E\left[e^{\sum_{i=-n}^{-m-1} g(\omega_{X_i}, Z_{i+1})} \middle| X_{-m} = x\right] = 0, \forall x \in \mathbb{Z}^d \text{ with } x \cdot \hat{u} = -m\right\}.$$

Explanation: Each of the conditional expectations must go to zero in order for W_n^- to go to zero as it was written as a finite sum of positive numbers times the conditional expectations. Therefore, $\{W_\infty^- = 0\}$ is measurable with respect to $\sigma(\omega_x : x \cdot \hat{u} \leq -m)$, in which the conditional expectations are measurable. $\square$

By Kolmogorov's 0-1 law, we have that $\mathbb{P}(W_\infty^- = 0) \in \{0,1\}$. By Fatou's lemma (Chapter 1, Theorem (3.5), page 16 of [4]) we have

$$\mathbb{E}\left[W_\infty\right] = \mathbb{E}\left[\liminf_{n\to\infty} W_n\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[W_n\right] = 1.$$

Similarly, $\mathbb{E}\left[W_\infty^-\right] \leq 1$. In other words, both W_∞ and W_∞^- are in $L^1(\mathbb{P})$. Also, observe that $\mathbb{W}_\infty$ and $\mathbb{W}_\infty^-$ are independent. Recall from Lemma 4 that when β is small enough, we have $\mathbb{P}(W_\infty > 0) = 1$. A similar fact holds for W_∞^-.

Lemma 8. *For $d \geq 4$, there exists $\beta_0 > 0$ such that for $\beta \in [0, \beta_0)$ we have $\mathbb{P}(W_\infty^- > 0) = 1$.*

Proof. Since we will be varying β in the course of this proof, we will bring it back to the notation. As in the proof of Lemma 4, we begin by a computation of

$$\begin{aligned}
\mathbb{E}[(W_n^-)^2] &= \mathbb{E}\Big[E \otimes E\Big[e^{\sum_{i=-n}^{-1}(\beta g(\omega_{X_i}, Z_{i+1}) - \lambda(\beta))} e^{\sum_{i=-n}^{-1}(\beta g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1}) - \lambda(\beta))}\Big]\Big] \\
&= E \otimes E\Big[\prod_{i=-n}^{-1} \mathbb{E}\big[e^{\beta g(\omega_{X_i}, Z_{i+1}) - \lambda(\beta)} e^{\beta g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1}) - \lambda(\beta)}\big]\Big] \\
&= E \otimes E\Big[\prod_{i=-n}^{-1} \frac{q^{(2)}((X_i, \tilde{X}_i), (X_{i+1}, \tilde{X}_{i+1}))}{p(Z_{i+1}) p(\tilde{Z}_{i+1})} \kappa(\beta)^{\mathbf{1}\{X_i = \tilde{X}_i\}}\Big] \\
&= \sum_{x, \tilde{x}} E^{(2)}\Big[\kappa(\beta)^{\sum_{i=0}^{n-1} \mathbf{1}\{X_i = \tilde{X}_i\}} \mathbf{1}\{X_n = \tilde{X}_n = 0\} \,\Big|\, X_0 = x, \tilde{X}_0 = \tilde{x}\Big] \\
&= \sum_{x, \tilde{x}} E^{(2)}\Big[\kappa(\beta)^{\sum_{i=0}^{n-1} \mathbf{1}\{X_i = \tilde{X}_i\}} \mathbf{1}\{X_n = \tilde{X}_n = -\tilde{x}\} \,\Big|\, X_0 = x - \tilde{x}, \tilde{X}_0 = 0\Big] \\
&= \sum_{y, \tilde{x}} E^{(2)}\Big[\kappa(\beta)^{\sum_{i=0}^{n-1} \mathbf{1}\{X_i = \tilde{X}_i\}} \mathbf{1}\{X_n = \tilde{X}_n = -\tilde{x}\} \,\Big|\, X_0 = y, \tilde{X}_0 = 0\Big] \\
&= \sum_{y} E^{(2)}\Big[\kappa(\beta)^{\sum_{i=0}^{n-1} \mathbf{1}\{X_i = \tilde{X}_i\}} \mathbf{1}\{X_n = \tilde{X}_n\} \,\Big|\, X_0 = y, \tilde{X}_0 = 0\Big] \\
&= \sum_{y} E^{\overline{Q}}\Big[\kappa(\beta)^{\sum_{i=0}^{n-1} \mathbf{1}\{Y_i = 0\}} \mathbf{1}\{Y_n = 0\} \,\Big|\, Y_0 = y\Big].
\end{aligned} \tag{3.1}$$

Here, Y_n under $\overline{Q}$ has the distribution of $X_n - \tilde{X}_n$ under $Q^{(2)}$. As we have seen in Lemma 4, $\overline{Q}$ is a Markov chain with transition kernel

$$\bar{q}(u, u+v) = \begin{cases} \sum_{z - \tilde{z} = v} q(z) q(\tilde{z}) & \text{if } u \neq 0, \\ \sum_{z - \tilde{z} = v} q((0,0), (z, \tilde{z})) & \text{if } u = 0. \end{cases}$$

We now need to tweak the computation we did for Lemma 4 a little bit. As the Y_n in the above computation starts at y and ends at 0, we will next reverse the process and have it instead start at 0 and end at y.

Random walk Y_n lives on the set

$$\mathbb{V} = \{y \in \mathbb{Z}^d : y \cdot \hat{u} = 0\}$$

and has increments in the set

$$\mathcal{N} = \{z - \tilde{z} : z, \tilde{z} \in \mathcal{R}\} \subset \mathbb{V}.$$

Define, for $u \in \mathbb{V}$ and $v \in \mathcal{N}$ the probability transition kernel

$$\hat{q}(u, u - v) = \frac{\bar{q}(u - v, u)}{\gamma(u)},$$

where

$$\gamma(u) = \sum_{v \in \mathcal{N}} \bar{q}(u - v, u).$$

Let $\widehat{Q}$ be the distribution of the Markov chain $\widehat{Y}_n$ using this kernel and started at $\widehat{Y}_0 = 0$.

Observe that if $u \notin \mathcal{N}$, then $u - v \neq 0$ for any $v \in \mathcal{N}$ and thus

$$\gamma(u) = \sum_{v \in \mathcal{N}} \sum_{z - \tilde{z} = v} q(z)q(\tilde{z}) = \sum_{z, \tilde{z}} q(z)q(\tilde{z}) = 1.$$

Therefore, for all $u \in \mathbb{V}$ and $v \in \mathcal{N}$, we have

$$\bar{q}(u, u + v) = \gamma(u)^{\mathbf{1}\{u \in \mathcal{N}\}} \hat{q}(u + v, u).$$

Continuing from (3.1), we have

$$\begin{aligned}
\mathbb{E}[(W_n^-)^2] &= \sum_y E^{\widehat{Q}}\Big[\kappa(\beta)^{\sum_{i=1}^n \mathbf{1}\{\widehat{Y}_i = 0\}} \prod_{i=0}^{n-1} \gamma(\widehat{Y}_i)^{\mathbf{1}\{\widehat{Y}_i \in \mathcal{N}\}} \mathbf{1}\{\widehat{Y}_n = y\}\Big] \\
&= E^{\widehat{Q}}\Big[\kappa(\beta)^{\sum_{i=1}^n \mathbf{1}\{\widehat{Y}_i = 0\}} \prod_{i=0}^{n-1} \gamma(\widehat{Y}_i)^{\mathbf{1}\{\widehat{Y}_i \in \mathcal{N}\}}\Big] \\
&\leq E^{\widehat{Q}}\Big[(\kappa(\beta)\bar{\gamma}(\beta))^{\sum_{i=0}^n \mathbf{1}\{\widehat{Y}_i \in \mathcal{N}\}}\Big], \\
&\leq E^{\widehat{Q}}\Big[(\kappa(\beta)\bar{\gamma}(\beta))^{\sum_{i=0}^\infty \mathbf{1}\{\widehat{Y}_i \in \mathcal{N}\}}\Big],
\end{aligned}$$

where we have used the fact that $\kappa(\beta) > 1$ and defined

$$\bar{\gamma}(\beta) = \max\{1, \max_{u \in \mathcal{N}} \gamma(u)\}.$$

If we can choose β small enough so that

$$E^{\widehat{Q}}\left[(\kappa(\beta)\bar{\gamma}(\beta))^{\sum_{i=0}^{\infty}\mathbf{1}\{\widehat{Y}_i\in\mathcal{N}\}}\right]<\infty \tag{3.2}$$

then we would have that W_n^- is uniformly integrable. This and the $\mathbb{P}$-almost sure convergence of W_n^- to W_∞^- imply the convergence also holds in $L^1(\mathbb{P})$. Then $1=\mathbb{E}[W_n^-]\to\mathbb{E}[W_\infty^-]$ showing that $\mathbb{P}(W_\infty^->0)>0$ and thus $\mathbb{P}(W_\infty^->0)=1$ as claimed in the lemma.

It remains to show (3.2). For this, note that as $\beta\to 0$ we have $\kappa(\beta)\to 1$ and also $\bar{\gamma}(\beta)\to 1$. Indeed, recall that we have already observed during the proof of Lemma 4 that as $\beta\to 0$ measure $\overline{Q}\to\overline{P}$, the difference of two independent P-random walks. Consequently, for any $u\in\mathbb{V}$

$$\gamma(u)=\sum_{v\in\mathcal{N}}\bar{q}(u-v,u)\underset{\beta\to 0}{\longrightarrow}\sum_{v\in\mathcal{N}}\sum_{z-\tilde{z}=v}p(z)p(\tilde{z})=\sum_{z,\tilde{z}}p(z)p(\tilde{z})=1.$$

Now, the remaining computation is similar to the one in Lemma 4. Namely, as long as there is a positive $\overline{P}$-probability of leaving the set $\mathcal{N}$, we see that at the limit $\beta\to 0$, the $\overline{P}$-probabililty of ever returning to set $\mathcal{N}$ is strictly less than one (when $d\geq 4$ and thus $\overline{P}$ is at least three-dimensional). Since $\kappa(\beta)\gamma(\beta)\to 1$ as $\beta\to 0$, we see that for β small enough, (3.2) must hold.

□

When $\mathbb{P}(W_\infty>0)=\mathbb{P}(W_\infty^->0)=1$, (e.g., for β small) define the probability measure $\mathbb{P}_\infty$ by its expectation $\mathbb{E}_\infty$ given by

$$\mathbb{E}_\infty[f]=\frac{\mathbb{E}\left[W_\infty^-W_\infty f\right]}{\mathbb{E}\left[W_\infty^-\right]\mathbb{E}\left[W_\infty\right]},$$

for a bounded measurable function $f(\omega)$.

Define the shift operator S on the space $\Omega\times\mathcal{R}^{\mathbb{N}}$ by:

$$S(\omega,z_{1,\infty})=(T_{z_1}\omega,z_{2,\infty}).$$

Theorem 3. *$\mathbb{P}_\infty(d\omega)\mu^\omega(dZ_{-\infty,\infty})$ is S-invariant.*

Proof. Consider a bounded function $f(\omega, Z_{1,m})$. Then

$$
\begin{aligned}
&\mathbb{E}_\infty E^{\mu^\omega}\left[f \circ S\right] \\
&= c\mathbb{E}\left[W_\infty^- W_\infty E^{\mu^\omega}\left[f(T_{Z_1}\omega, Z_{2,m+1})\right]\right] \\
&= c\mathbb{E}\left[W_\infty^- W_\infty E\left[e^{\sum_{i=0}^m g(\omega_{X_i}, Z_{i+1})} \frac{W_\infty\left(T_{X_{m+1}}\omega\right)}{W_\infty} f\left(T_{Z_1}\omega, Z_{2,m+1}\right)\right]\right] \\
&= c\sum_{z\in\mathcal{R}} \mathbb{E}\left[W_\infty^- e^{g(\omega_0, z)} p(z) E\left[e^{\sum_{i=1}^m g(\omega_{X_i}, Z_{i+1})} W_\infty\left(T_{X_{m+1}}\omega\right) f\left(T_z\omega, Z_{2,m+1}\right) \middle| Z_1 = z\right]\right] \\
&= c\sum_{z\in\mathcal{R}} \mathbb{E}\left[W_\infty^-(T_{-z}\omega) e^{g(\omega_{-z}, z)} p(z) E\left[e^{\sum_{i=1}^m g(\omega_{X_i - z}, Z_{i+1})} W_\infty\left(T_{X_m - z}\omega\right) f\left(\omega, Z_{2,m+1}\right) \middle| Z_1 = z\right]\right] \\
&= c\sum_{z\in\mathcal{R}} \mathbb{E}\left[W_\infty^-(T_{-z}\omega) e^{g(\omega_{-z}, z)} p(z) E\left[e^{\sum_{i=0}^{m-1} g(\omega_{X_i}, Z_{i+1})} W_\infty\left(T_{X_m}\omega\right) f\left(\omega, Z_{1,m}\right)\right]\right] \\
&= c\mathbb{E}\left[\sum_{z\in\mathcal{R}} W_\infty^-(T_{-z}\omega) e^{g(\omega_{-z}, z)} p(z) W_\infty E^{\mu^\omega}\left[f\right]\right] \\
&= c\mathbb{E}\left[W_\infty^- W_\infty E^{\mu^\omega}\left[f\right]\right] \\
&= \mathbb{E}_\infty E^{\mu^\omega}\left[f\right].
\end{aligned}
$$

Here, $c = 1/(\mathbb{E}[W_\infty^-]\mathbb{E}[W_\infty])$. In the third equality, we decomposed the E expectation according to the possible values of Z_1. In the fourth equality, we used the shift-invariance of $\mathbb{P}$. In the second to last equality, we used the fact that

$$\sum_z W_\infty^-(T_{-z}\omega) e^{g(\omega_{-z}, z)} p(z) = W_\infty^-(\omega), \mathbb{P} - \text{a.s.}$$

This fact comes from the way W_∞^- is defined. Indeed:

$$\sum_z W_n^-(T_{-z}\omega) e^{g(\omega_{-z}, z)} p(z) = E\left[e^{\sum_{i=n-1}^0 g(\omega_{X_i}, Z_{i+1})}\right] = W_{n+1}^-(\omega).$$

Take $n \to \infty$. The theorem is proved. $\square$

Theorem 4. *If* $\mathbb{P}(W_\infty > 0) = \mathbb{P}(W_\infty^- > 0) = 1$, *then* $\mathbb{E}_\infty \mu^\omega = Q$.

Proof. Let $f : \mathcal{R}^m \to \mathbb{R}$ be a bounded function. Then

$$
\begin{aligned}
\mathbb{E}_\infty E^{\mu^\omega}[f] &= \frac{\mathbb{E}\left[W_\infty^- E\left[e^{\sum_{i=0}^{m-1} g(\omega_{X_i}, Z_{i+1})} W_\infty(T_{X_m}\omega) f(Z_{1,m})\right]\right]}{\mathbb{E}[W_\infty^-]\mathbb{E}[W_\infty]} \\
&= \frac{E\left[\mathbb{E}\left[W_\infty^- e^{\sum_{i=0}^{m-1} g(\omega_{X_i}, Z_{i+1})} W_\infty(T_{X_m}\omega)\right] f(Z_{1,m})\right]}{\mathbb{E}[W_\infty^-]\mathbb{E}[W_\infty]}.
\end{aligned}
$$

Since $W_\infty(T_{X_m}\omega), W_\infty^-$, and $e^{\sum_{i=0}^{m-1} g(\omega_{X_i}, Z_{i+1})}$ are independent, and $\mathbb{P}$ is translation invariant, we can factor $\mathbb{E}[W_\infty^-]$ and $\mathbb{E}[W_\infty]$ out to get

$$\mathbb{E}_\infty E^{\mu^\omega}[f] = E\left[f(Z_{1,m}) \prod_{i=0}^{m-1} \mathbb{E}\left[e^{g(\omega_0, Z_{i+1})} \right] \right] = E^Q[f]. \qquad \square$$

As a corollary, the next proposition shows that the averaged process is in fact equivalent to the annealed one.

Proposition 1. *Assume* $\mathbb{P}(W_\infty^- > 0) = \mathbb{P}(W_\infty > 0) = 1$*. Then* $Q \ll \mathbb{E}\mu^\omega \ll Q$*.*

Proof. Take $A \in \mathcal{F}$. Note that W_∞ and $\mu^\omega(A)$ are $\sigma(\omega_x : x \cdot \hat{u} \geq 0)$-measurable and W_∞^- is $\sigma(\omega_x : x \cdot \hat{u} < 0)$-measurable. Then we have

$$\begin{aligned} Q(A) = 0 &\iff \mathbb{E}_\infty \mu^\omega(A) = 0 \\ &\iff \mathbb{E}\left[W_\infty^- W_\infty \, \mu^\omega(A) \right] = 0 \\ &\iff \mathbb{E}[W_\infty \mu^\omega(A)] = 0 \\ &\iff \mu^\omega(A) = 0, \mathbb{P} - \text{a.s.} \\ &\iff \mathbb{E}\mu^\omega(A) = 0. \end{aligned}$$

The proposition is proved. $\square$

Consequently, we can deduce that an invariance principle holds under the averaged measure $\mathbb{E}\mu^\omega$. However, we are interested in fluctuations under the quenched measure μ^ω. This will require us to study $\mathbb{E}\mu^\omega \otimes \mu^\omega$.

CHAPTER 4

TWO INDEPENDENT WALKS IN A COMMON ENVIRONMENT

In order to obtain a central limit theorem for X_n under μ^ω, we need to study the behavior of two independent copies of X_n, evolving in the same environment ω.

Recall $Q^{(2)}$, the Markov chain $(X_n, \tilde{X}_n)$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ starting at $(0,0)$ with kernel

$$q^{(2)}\left((x,\tilde{x}),(x+z,\tilde{x}+\tilde{z})\right) = \begin{cases} q(z)q(\tilde{z}) & \text{if } x \neq \tilde{x}, z, \tilde{z} \in \mathcal{R} \\ \kappa^{-1}\mathbb{E}\left[e^{g(\omega_0,z)}e^{g(\omega_0,\tilde{z})}\right] p(z)p(\tilde{z}) & \text{if } x = \tilde{x}, z, \tilde{z} \in \mathcal{R}, \end{cases}$$

where $\kappa = E \otimes E\left[\mathbb{E}\left[e^{g(\omega_0,Z_1)}e^{g(\omega_0,\tilde{Z}_1)}\right]\right] = \mathbb{E}\left[\left(E\left[e^{g(\omega_0,Z_1)}\right]\right)^2\right]$.

Theorem 5. *Assume $d \geq 4$ and weak disorder holds. Then*

$$\mathbb{E}\mu^\omega \otimes \mu^\omega \ll Q^{(2)}.$$

Proof. Take $A \in \mathcal{F} \otimes \mathcal{F}$ such that $Q^{(2)}(A) = 0$. Recall that the distribution $\overline{Q}$ of $X_n - \tilde{X}_n$, induced by $Q^{(2)}$, is itself a Markov chain with kernel

$$\overline{q}(u, u+v) = \begin{cases} \sum_{z-\tilde{z}=v} q(z)q(\tilde{z}) & \text{if } u \neq 0 \\ \sum_{z-\tilde{z}=v} q((0,0),(z,\tilde{z})) & \text{if } u = 0. \end{cases}$$

Let τ_0 be the time of first return of this chain to 0. Then

$$\overline{Q}(\tau_0 < \infty) = \overline{q}(0,0) + \sum_{v \neq 0} \overline{q}(0,v)\overline{Q}_v(\tau_0 < \infty) < 1,$$

where $\overline{Q}_v$ denotes the Markov chain started at v. Notably $\overline{Q}_v(\tau_0 < \infty)$ is the same as for the difference of two independent Q-random walks. Thus when $d \geq 4$, this quantity is strictly less than 1. Since $\exists v \neq 0 : \overline{q}(0,v) > 0$ we have

$$\sum_{v \neq 0} \overline{q}(0,v)\overline{Q}_v(\tau_0 < \infty) < \sum_{v \neq 0} \overline{q}(0,v) = 1 - \overline{q}(0,0).$$

This implies $\overline{Q}(\tau_0 < \infty) < 1$. Consequently, $Q^{(2)}(I_\infty < \infty) = 1$ and $E^{(2)}\left[\kappa^{I_\infty}\mathbf{1}_A\right] = 0$. (We denote expectation under $Q^{(2)}$ by $E^{(2)}$.) Consider the measure $Q^{(2)}_\infty$ such that $\frac{dQ^{(2)}_\infty}{dQ^{(2)}} = \kappa^{I_\infty}$.

Although $Q_\infty^{(2)}\left(\mathcal{R}^{\mathbb{N}}\times\mathcal{R}^{\mathbb{N}}\right) = E^{(2)}\left[\kappa^{I_\infty}\right] = \mathbb{E}\left[W_\infty^2\right]$ could be infinite even under weak disorder, $Q_\infty^{(2)}$ is σ-finite. Indeed,

$$Q_\infty^{(2)}\left(I_\infty \le \ell\right) = E^{(2)}\left[\kappa^{I_\infty}\mathbf{1}\{I_\infty \le \ell\}\right] \le \kappa^\ell < \infty$$

and as $\ell \to \infty$, $\{I_\infty \le \ell\}$ increases to $\{I_\infty < \infty\}$ which has full $Q^{(2)}$-measure. By Lemma (3.1) of Appendix A.3 of [4] (page 452), there exist sets $A_{m,k} \in \mathcal{F}_m \otimes \mathcal{F}_m$ such that

i) for all k, $A_{m,k}$ increases with m,

ii) $\cup_m A_{m,k}$ decreases with k,

iii) $\cup_m A_{m,k} \supset A$ for all k, and

iv) $Q_\infty^{(2)}\left(\cup_m A_{m,k}\right) \searrow Q_\infty^{(2)}(A) = 0$ as $k \to \infty$.

By iv), for each ℓ, there exists k_ℓ such that

$$Q_\infty^{(2)}\left(\cup_m A_{m,k_\ell}\right) \le \frac{1}{2\ell}.$$

By i) and the monotone convergence theorem (Chapter 1, Theorem (3.6), page 16 of [4]), both $Q_\infty^{(2)}(A_{m,k}) \nearrow Q_\infty^{(2)}\left(\cup_m A_{m,k}\right)$ and $\mathbb{E}\mu^\omega \otimes \mu^\omega(A_{m,k}) \nearrow \mathbb{E}\mu^\omega \otimes \mu^\omega(\cup_m A_{m,k})$ as $m \to \infty$. Thus there exists m_ℓ such that

$$Q_\infty^{(2)}\left(A_{m_\ell,k_\ell}\right) \le \frac{1}{\ell}$$

and

$$\mathbb{E}\mu^\omega \otimes \mu^\omega\left(\cup_m A_{m,k_\ell}\right) \le \mathbb{E}\mu^\omega \otimes \mu^\omega\left(A_{m_\ell,k_\ell}\right) + \frac{1}{\ell}.$$

So for $A_{m_\ell,k_\ell} \in \mathcal{F}_{m_\ell} \otimes \mathcal{F}_{m_\ell}$, $Q_\infty^{(2)}\left(A_{m_\ell,k_\ell}\right) \to 0$ as $\ell \to \infty$. We now need a lemma.

Lemma 9. *Let $A_m \in \mathcal{F}_m \otimes \mathcal{F}_m, m \ge 0$, such that $Q_\infty^{(2)}(A_m) \to 0$ as $m \to \infty$. Then $\mathbb{E}\mu^\omega \otimes \mu^\omega(A_m) \to 0$.*

By Lemma 9, $\mathbb{E}\mu^\omega \otimes \mu^\omega(A_{m_\ell,k_\ell}) \to 0$ as $\ell \to \infty$. But then by iii),

$$\mathbb{E}\mu^\omega \otimes \mu^\omega(A) \le \mathbb{E}\mu^\omega \otimes \mu^\omega\left(\cup_m A_{m,k_\ell}\right) \le \mathbb{E}\mu^\omega \otimes \mu^\omega\left(A_{m_\ell,k_\ell}\right) + \frac{1}{\ell}.$$

To complete the proof, take $\ell \to \infty$ so that the right two terms both go to 0. $\mathbb{E}\mu^\omega \otimes \mu^\omega(A)$ does not depend on ℓ, so $\mathbb{E}\mu^\omega \otimes \mu^\omega(A) = 0$. $\square$

Proof of Lemma 9. It is enough to show $\mu^\omega \otimes \mu^\omega(A_m) \to 0$ as $m \to \infty$ in probability. To this end, compute,

$$\begin{aligned}\mu^\omega \otimes \mu^\omega(A_m) &= \lim_{n\to\infty} \mu_n^\omega \otimes \mu_n^\omega(A_m) \\ &= \lim_{n\to\infty} \frac{E \otimes E\left[\mathbf{1}_{A_m}(X_{0,m}, \tilde{X}_{0,m}) e^{\sum_{i=0}^{n-1} g(\omega_{X_i}, Z_{i+1})} e^{\sum_{i=0}^{n-1} g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1})}\right]}{W_n^2} \\ &= \frac{\lim_{n\to\infty} E \otimes E\left[\mathbf{1}_{A_m}(X_{0,m}, \tilde{X}_{0,m}) e^{\sum_{i=0}^{n-1} g(\omega_{X_i}, Z_{i+1})} e^{\sum_{i=0}^{n-1} g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1})}\right]}{W_\infty^2}.\end{aligned}$$

We are allowed to take the limit into the numerator because $W_n^2 \to W_\infty^2$ converges $\mathbb{P}$-almost surely. Multiplying by W_∞^2 and taking expectation $\mathbb{E}$, we have

$$\mathbb{E}\left[W_\infty^2 \mu^\omega \otimes \mu^\omega(A_m)\right] = \mathbb{E}\left[\lim_{n\to\infty} E \otimes E\left[\mathbf{1}_{A_m}(X_{0,m}, \tilde{X}_{0,m}) e^{\sum_{i=0}^{n-1} g(\omega_{X_i}, Z_{i+1})} e^{\sum_{i=0}^{n-1} g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1})}\right]\right].$$

With $E \otimes E\left[\mathbf{1}_{A_m}(X_{0,m}, \tilde{X}_{0,m}) e^{\sum_{i=0}^{n-1} g(\omega_{X_i}, Z_{i+1})} e^{\sum_{i=0}^{n-1} g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1})}\right] \geq 0$ and its limit equal to its $\liminf$, apply Fatou's lemma to get

$$\begin{aligned}&\mathbb{E}\left[\lim_{n\to\infty} E \otimes E\left[\mathbf{1}_{A_m}(X_{0,m}, \tilde{X}_{0,m}) e^{\sum_{i=0}^{n-1} g(\omega_{X_i}, Z_{i+1})} e^{\sum_{i=0}^{n-1} g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1})}\right]\right] \\ &= \mathbb{E}\left[\liminf_{n\to\infty} E \otimes E\left[\mathbf{1}_{A_m}(X_{0,m}, \tilde{X}_{0,m}) e^{\sum_{i=0}^{n-1} g(\omega_{X_i}, Z_{i+1})} e^{\sum_{i=0}^{n-1} g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1})}\right]\right] \\ &\leq \liminf_{n\to\infty} \mathbb{E}\left[E \otimes E\left[\mathbf{1}_{A_m}(X_{0,m}, \tilde{X}_{0,m}) e^{\sum_{i=0}^{n-1} g(\omega_{X_i}, Z_{i+1})} e^{\sum_{i=0}^{n-1} g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1})}\right]\right].\end{aligned}$$

Because quantities are positive, by Fubini's theorem (Appendix A.6, Theorem (6.2), page 470 of [4]),

$$\begin{aligned}&\liminf_{n\to\infty} \mathbb{E}\left[E \otimes E\left[\mathbf{1}_{A_m}(X_{0,m}, \tilde{X}_{0,m}) e^{\sum_{i=0}^{n-1} g(\omega_{X_i}, Z_{i+1})} e^{\sum_{i=0}^{n-1} g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1})}\right]\right] \\ &\qquad = \liminf_{n\to\infty} E \otimes E\left[\mathbf{1}_{A_m} \mathbb{E}\left[e^{\sum_{i=0}^{n-1} g(\omega_{X_i}, Z_{i+1})} e^{\sum_{i=0}^{n-1} g(\omega_{\tilde{X}_i}, \tilde{Z}_{i+1})}\right]\right].\end{aligned}$$

Recall

$$\mathbb{E}\left[e^{g(\omega_x, z)} e^{g(\omega_{\tilde{x}}, \tilde{z})}\right] = \begin{cases} \mathbb{E}\left[e^{g(\omega_0, z)}\right] \mathbb{E}\left[e^{g(\omega_0, \tilde{z})}\right] = \frac{q^{(2)}((x,\tilde{x}),(x+z,\tilde{x}+\tilde{z}))}{p(z)p(\tilde{z})} & \text{if } x \neq \tilde{x}, \\ \mathbb{E}\left[e^{g(\omega_0, z)} e^{g(\omega_0, \tilde{z})}\right] = \frac{\kappa q^{(2)}((x,\tilde{x}),(x+z,\tilde{x}+\tilde{z}))}{p(z)p(\tilde{z})} & \text{if } x = \tilde{x}. \end{cases}$$

Replacing $E \times E$ with $E^{(2)}$,

$$\begin{aligned}
&\liminf_{n\to\infty} E\otimes E\left[\mathbf{1}_{A_m}\mathbb{E}\left[e^{\sum_{i=0}^{n-1} g(\omega_{X_i},Z_{i+1})}e^{\sum_{i=0}^{n-1} g(\omega_{\tilde{X}_i},\tilde{Z}_{i+1})}\right]\right]\\
&\qquad=\liminf_{n\to\infty} E\otimes E\left[\mathbf{1}_{A_m}\prod_{i=0}^{n-1}\mathbb{E}\left[e^{g(\omega_{X_i},Z_{i+1})}e^{g(\omega_{\tilde{X}_i},\tilde{Z}_{i+1})}\right]\right]\\
&\qquad=\lim_{n\to\infty} E^{(2)}\left[\mathbf{1}_{A_m}\kappa^{\sum_{i=0}^{n-1}\mathbf{1}\{X_i=\tilde{X}_i\}}\right]\\
&\qquad=E^{(2)}\left[\mathbf{1}_{A_m}\kappa^{I_\infty}\right]\\
&\qquad=Q_\infty^{(2)}(A_m).
\end{aligned}$$

(Recall $I_\infty = \sum_{i=0}^\infty \mathbf{1}\{X_i = \tilde{X}_i\}$.) The third equation is justified by the monotone convergence theorem (Chapter 1, Theorem (3.5), page 16 of [4]). In summary,

$$\mathbb{E}\left[W_\infty^2\mu^\omega\otimes\mu^\omega(A_m)\right]\le Q_\infty^{(2)}(A_m)\to 0 \text{ as } m\to\infty.$$

So $W_\infty^2\mu^\omega\otimes\mu^\omega(A_m)\to 0$ in $\mathbb{P}$-probability and so does $\mu^\omega\otimes\mu^\omega(A_m)\to 0$ as $m\to\infty$. The claim now follows from the bounded convergence theorem ([4] 17). □

Lemma 10. *When $d\ge 4$, $Q^{(2)}$ and $Q\otimes Q$ are mutually absolutely continuous:*

$$Q\otimes Q\ll Q^{(2)}\ll Q\otimes Q.$$

Proof. The Radon-Nikodym derivative of the restrictions to $\mathcal{F}_m\otimes\mathcal{F}_m$ is

$$\frac{dQ^{(2)}|_{\mathcal{F}_m\otimes\mathcal{F}_m}}{dQ\otimes Q|_{\mathcal{F}_m\otimes\mathcal{F}_m}}(x_{0,m},\tilde{x}_{0,m})=\prod_{i=0}^{m-1}\frac{q^{(2)}\left((x_i,\tilde{x}_i),(x_{i+1},\tilde{x}_{i+1})\right)}{q(z_{i+1})q(\tilde{z}_{i+1})}.$$

Recall:

$$\frac{q^{(2)}((x,\tilde{x}),(x+z,\tilde{x}+\tilde{z}))}{q(z)q(\tilde{z})}=\begin{cases}1 & \text{if } x\ne\tilde{x},\\ \frac{\kappa^{-1}\mathbb{E}[e^{g(\omega_0,z)}e^{g(\omega_0,\tilde{z})}]p(z)p(\tilde{z})}{\mathbb{E}[e^{g(\omega_0,z)}]p(z)\mathbb{E}[e^{g(\omega_0,\tilde{z})}]p(\tilde{z})} & \text{if } x=\tilde{x}.\end{cases}$$

The product merely takes note of when the fraction is not 1. Hence

$$\prod_{i=0}^{m-1}\frac{q^{(2)}\left((x_i,\tilde{x}_i),(x_{i+1},\tilde{x}_{i+1})\right)}{q(z_{i+1})q(\tilde{z}_{i+1})}=\prod_{i=0}^{m-1}\left(\frac{\mathbb{E}[e^{g(\omega_0,z_{i+1})}e^{g(\omega_0,\tilde{z}_{i+1})}]}{\kappa\mathbb{E}[e^{g(\omega_0,z_{i+1})}]\mathbb{E}[e^{g(\omega_0,\tilde{z}_{i+1})}]}\right)^{\mathbf{1}\{x_i=\tilde{x}_i\}}. \tag{4.1}$$

Since $Q\otimes Q(I_\infty<\infty)=Q^{(2)}(I_\infty<\infty)=1$, when $d\ge 4$, we have that $X_i=\tilde{X}_i$ only for finitely many i, both $Q\otimes Q$- and $Q^{(2)}$-almost surely. The limit of (4.1) is then some random positive finite number. The unrestricted Radon-Nikodym derivative is thus

$$\frac{dQ^{(2)}}{dQ\otimes Q}(x_{0,\infty},\tilde{x}_{0,\infty})=\prod_{i=0}^{\infty}\left(\frac{\mathbb{E}[e^{g(\omega_0,z_{i+1})}e^{g(\omega_0,\tilde{z}_{i+1})}]}{\kappa\mathbb{E}[e^{g(\omega_0,z_{i+1})}]\mathbb{E}[e^{g(\omega_0,\tilde{z}_{i+1})}]}\right)^{\mathbf{1}\{x_i=\tilde{x}_i\}}\in(0,\infty).$$

The lemma is proved. □

Corollary 1. *When $d \geq 4$ and weak disorder holds, we have*

$$\mathbb{E}\mu^\omega \otimes \mu^\omega \ll Q \otimes Q.$$

If furthermore $\mathbb{P}(W_\infty^- > 0) = 1$, then

$$\mathbb{E}_\infty \mu^\omega \otimes \mu^\omega \sim \mathbb{E}\mu^\omega \otimes \mu^\omega$$

and so

$$\mathbb{E}_\infty \mu^\omega \otimes \mu^\omega \ll Q \otimes Q.$$

Proof. The first claim comes from combining Theorem 5 and Lemma 10. For the second claim, write

$$\begin{aligned} \mathbb{E}\mu^\omega \otimes \mu^\omega(A) = 0 &\Leftrightarrow \mu^\omega \otimes \mu^\omega(A) = 0, \ \mathbb{P}\text{-a.s.} \\ &\Leftrightarrow W_\infty^- W_\infty \mu^\omega \otimes \mu^\omega(A) = 0, \ \mathbb{P}\text{- a.s.} \\ &\Leftrightarrow \mathbb{E}\left[W_\infty^- W_\infty \mu^\omega \otimes \mu^\omega(A)\right] = 0, \\ &\Leftrightarrow \mathbb{E}_\infty \mu^\omega \otimes \mu^\omega(A) = 0. \end{aligned}$$

This proves the corollary. $\square$

CHAPTER 5

INVARIANCE PRINCIPLE

We are now ready to prove a version of Theorem 1 for μ^ω.

Theorem 6. *Assume $d \geq 4$ and $\mathbb{P}(W_\infty > 0) = 1$. Then $\forall F \in C_b(\mathbb{W})$,*

$$E^{\mu^\omega}[F(B_n)] \to \mathbf{E}[F]$$

in $\mathbb{P}$-probability.

Proof. Define the continuous, bounded function,

$$G(B, \overline{B}) = (F(B) - \mathbf{E}[F])(F(\overline{B}) - \mathbf{E}[F]) \in C_b(\mathbb{W} \times \mathbb{W}).$$

Take any $\nu \ll Q \otimes Q$. Let $a_n = E^\nu \left[G(B_n, \overline{B}_n)\right]$ and take a convergent subsequence $a_{n_j} \to b$. By the almost sure central limit theorem for sums of i.i.d. random variables (Theorem 2 of [7])

$$\frac{1}{\log N} \sum_{j=1}^{N} \frac{1}{j} G(B_{n_j}, \overline{B}_{n_j}) \to 0, Q \otimes Q\text{-a.s. as } N \to \infty. \tag{5.1}$$

The same holds ν-a.s. By bounded convergence,

$$\frac{1}{\log N} \sum_{j=1}^{N} \frac{1}{j} a_{n_j} \to 0 \text{ as } N \to \infty.$$

So it must be that $b = 0$ is the only limit point, i.e., $E^\nu[G(B_n, \overline{B}_n)] \to 0$ as $n \to \infty$. Apply this to $\nu = \mathbb{E}\mu^\omega \otimes \mu^\omega$ and take $n \to \infty$ to get

$$\mathbb{E}\left[E^{\mu^\omega} \otimes E^{\mu^\omega}\left[G(B_n, \overline{B}_n)\right]\right] \to 0. \tag{5.2}$$

But

$$E^{\mu^\omega} \otimes E^{\mu^\omega}\left[G(B_n, \overline{B}_n)\right] = (E^{\mu^\omega}[F(B_n) - \mathbf{E}[F]])^2$$

and so (5.2) implies that

$$E^{\mu^\omega}[F(B_n)] \to \mathbf{E}[F] \text{ in } \mathbb{P}\text{-probability}.$$

The theorem is proved. $\square$

Remark 5. *Say also* $\mathbb{P}(W_\infty^- > \infty) = 1$. *Apply the above argument with* $\nu = \mathbb{E}_\infty \mu^\omega \otimes \mu^\omega$ *to get the same conclusion:*

$$\lim_{n\to\infty} \mathbb{E}_\infty \left[\left(E^{\mu^\omega} \left[F(B_n) \right] - \mathbf{E} \left[F \right] \right)^2 \right] = 0. \tag{5.3}$$

For $i \in \{1, \ldots, d\}$ *take* $F(B) = B(1) \cdot e_i$ *(although* $B(1)$ *is not bounded.) Then* $\mathbf{E}\left[F\right] = 0$ *and*

$$F(B_n) = \frac{(X_n - nv) \cdot e_i}{\sqrt{q(e_i)(1 - q(e_i))n}}.$$

Recall that $Q = \mathbb{E}_\infty \mu^\omega$. *Thus*

$$nv = E^Q \left[X_n \right] = \mathbb{E}_\infty E^{\mu^\omega} \left[X_n \right]$$

and (5.3) *says*

$$\mathrm{Var}^{\mathbb{P}_\infty} \left(E^{\mu^\omega} \left[X_n \cdot e_i \right] \right) = o(n), \text{ for each } i \in \{1, \ldots, d\}.$$

If we improve this a bit to $\mathcal{O}(n^\alpha)$, *with* $\alpha < 1$, *then [13] would imply that Theorem 6 holds with "in* $\mathbb{P}$*-probability" replaced by "*$\mathbb{P}$*-almost surely." We in fact expect the variance to be* $\mathcal{O}(n^\varepsilon)$ *for any* $\varepsilon > 0$ *(when* $d \geq 4$*).*

Finally, we prove Theorem 1, which we restate for convenience.

Theorem 7. *Assume* $d \geq 4$ *and* $\mathbb{P}(W_\infty > 0) = 1$. *Then* $\forall F \in C_b(\mathbb{W})$,

$$E^{\mu_n^\omega}[F(B_n)] \to \mathbf{E}[F]$$

in $\mathbb{P}$*-probability.*

Proof. The proof follows word for word that of Proposition 4.3 of [3], page 1758. □

REFERENCES

[1] E. Bolthausen, *A note on the diffusion of directed polymers in a random environment*, Comm. Math. Phys., 123 (1989), pp. 529–534.

[2] F. Comets and N. Yoshida, *Brownian directed polymers in random environment*, Communications in Mathematical Physics, 254 (2005), pp. 257–287.

[3] ——, *Directed polymers in random environment are diffusive at weak disorder*, Ann. Probab., 34 (2006), pp. 1746–1770.

[4] R. Durrett, *Probability: theory and examples*, Duxbury Press, Belmont, CA, second ed., 1996.

[5] D. A. Huse and C. L. Henley, *Pinning and roughening of domain walls in ising systems due to random impurities*, Phys. Rev. Lett., 54 (1985), pp. 2708–2711.

[6] J. Z. Imbrie and T. Spencer, *Diffusion of directed polymers in a random environment*, Journal of Statistical Physics, 52 (1988), pp. 609–626.

[7] M. T. Lacey and W. Philipp, *A note on the almost sure central limit theorem*, Statistics and Probability Letters, 9 (1990), pp. 201 – 205.

[8] H. Lacoin, *New bounds for the free energy of directed polymers in dimension 1 + 1 and 1 + 2*, Communications in Mathematical Physics, 294 (2010), pp. 471–503.

[9] C. Licea, C. Newman, and M. Piza, *Superdiffusivity in first-passage percolation*, Probability Theory and Related Fields, 106 (1996), pp. 559–591.

[10] O. Mejane, *Upper bound of a volume exponent for directed polymers in a random environment*, Annales de l'Institut Henri Poincare (B) Probability and Statistics, 40 (2004), pp. 299 – 308.

[11] M. Petermann, *Superdiffusivity of Directed Polymers in the Random Environment*, Ph.D. thesis, University of Zurich, Zurich, Switzerland, 2000.

[12] M. S. T. Piza, *Directed polymers in a random environment: some results on fluctuations*, J. Statist. Phys., 89 (1997), pp. 581–603.

[13] F. Rassoul-Agha and T. Seppä'läinen, *An almost sure invariance principle for random walks in a space-time random environment*, ArXiv Mathematics e-prints, (2004).

[14] T. Seppüläinen, *Scaling for a one-dimensional directed polymer with boundary conditions*, Ann. Probab., 40 (2012), pp. 19–73.

[15] R. Song and X. Y. Zhou, *A remark on diffusion of directed polymers in random environments*, Journal of Statistical Physics, 85 (1996), pp. 277–289.

[16] M. V. WÜTHRICH, *Fluctuation results for Brownian motion in a Poissonian potential*, Ann. Inst. H. Poincaré Probab. Statist., 34 (1998), pp. 279–308.

[17] ———, *Superdiffusive behavior of two-dimensional Brownian motion in a Poissonian potential*, Ann. Probab., 26 (1998), pp. 1000–1015.